On a : $\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$

$= (n+1)^2(\frac{n^2}{4} + n + 1) = (n+1)^2(\frac{n^2+4n+4}{4}) = (n+1)^2 \frac{(n+2)^2}{4}$

Conclusion : $\forall n \in \mathbf{N}, \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

3°/ On a : $\forall x \in \mathbf{R}_+, x - \frac{x^3}{6} \leq \sin x \leq x$ et $\forall k \geq 1, \forall \alpha \geq 0, \frac{\alpha k}{n^2} \geq 0$

donc pour $x = \frac{\alpha k}{n^2}$ on a : $\frac{\alpha k}{n^2} - \frac{\alpha^3 k^3}{6n^6} \leq \sin(\frac{\alpha k}{n^2}) \leq \frac{\alpha k}{n^2} \Rightarrow$

$$\sum_{k=1}^{n}\left(\frac{\alpha\ k}{n^2} - \frac{\alpha^3\ k^3}{6n^6}\right) \leq U_n(\alpha) \leq \sum_{k=1}^{n}\frac{\alpha\ k}{n^2}$$

or : $\sum_{k=1}^{n}\left(\frac{\alpha\ k}{n^2} - \frac{\alpha^3\ k^3}{6n^6}\right) = \frac{\alpha}{n^2}\sum_{k=1}^{n} k - \frac{\alpha^3}{6n^6}\sum_{k=1}^{n} k^3$

$= \frac{\alpha}{n^2}\frac{n(n+1)}{2} - \frac{\alpha^3}{6n^6}\cdot\frac{n^2(n+1)^2}{4} = \frac{\alpha(n+1)}{2n} - \frac{\alpha^3(n+1)^2}{24n^4}$

d'où $\forall n \geq 1 : \frac{\alpha(n+1)}{2n} - \frac{\alpha^3(n+1)^2}{24n^4} \leq U_n(\alpha) \leq \alpha\frac{n+1}{2n}$.

4°/ On a : $\forall n \geq 1 : \frac{\alpha(n+1)}{2n} - \frac{\alpha^3(n+1)^2}{24n^4} \leq U_n(\alpha) \leq \alpha\frac{n+1}{2n}$

et $\lim_{n\to+\infty}(\frac{\alpha(n+1)}{2n} - \frac{\alpha^3(n+1)^2}{24n^4}) = \lim_{n\to+\infty}\alpha\frac{n+1}{2n} = \frac{\alpha}{2}$

d'où $\lim_{n\to+\infty} U_n(\alpha) = \frac{\alpha}{2}$.

5°/ On a : $\forall x \in \mathbf{R}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}$; $(i^2 = -1)$

$\Rightarrow \sin^3 x = -\frac{1}{8i}(e^{ix} - e^{-ix})^3 = -\frac{1}{8i}(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix})$

$\Rightarrow \sin^3 x = -\frac{1}{8i}(e^{3ix} - e^{-3ix} - 3(e^{ix} - e^{-ix}))$